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A GENERAL SOLUTION OF A LINEAR DISSIPATIVE OSCILLATORY SYSTEM AVOIDING DECOMPOSITION INTO EIGENVECTORS†

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An algorithm for constructing a general solution of an oscillatory system without determining the eigenvectors, which are the main source of the computational instability of classical methods, is proposed and justified. Souriau's algorithm [1–3], developed for solving systems of linear algebraic equations, is extended to the construction of a general solution of systems of linear ordinary differential equations. The adjoint matrix, consisting of the cofactors of the elements of the initial matrix, and the representation of the characteristic polynomial in terms of the characteristic adjoint matrix, play an important role here. The algorithm consists of simple algebraic operations, in addition to the numerical integration of a single differential equation, called the characteristic equation.

1. INTRODUCTION

Linear oscillations of mechanical systems are described by the second-order matrix linear ordinary differential equation

$$M\ddot{x} + C\dot{x} + Kx = F \tag{1.1}$$

Here x is an n-dimensional vector of the configuration variables, M, C and K are $n \times n$ matrices of the masses and the damping and stiffness coefficients, F are the external forces, and a dot denotes differentiation with respect to time.

The main difficulty of classical methods of constructing a general solution of such systems is calculating the eigenvectors. Small deviations of the matrix coefficients M, C and K can lead to considerable deviations of the components of the eigenvectors and to computational instability.

The basis of Souriau's algorithm [1–3], developed to solve systems of linear algebraic equations, is the idea of an adjoint matrix. An extension of this algorithm to construct a general solution of systems of linear ordinary differential equations is proposed. The advantage of this method is that it avoids the need to calculate eigenvalues and eigenvectors. Moreover, for a dissipative system it is not necessary to satisfy the specific properties of the matrix C [4] and symmetry, as well as the positiveness of the mass and stiffness matrices.

We will first give a brief description of Souriau's algorithm for linear algebraic equations, and we will then extend it to first-order linear matrix ordinary differential equations and then to mechanical systems described by second-order linear matrix ordinary differential equations. Finally, we will consider a system with external excitation.

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2. SOURIAU'S ALGORITHM

We will introduce some fundamental definitions and notation which will be required later. Suppose λ is a scalar and A is an $n \times n$ matrix, I is the identity matrix, adj A is the adjoint matrix, i.e. the transposed matrix of the cofactors of the elements of the matrix A, $\lambda I - A$ is the characteristic matrix, whose determinant is equal to the characteristic polynomial $P(\lambda)$, and its adjoint matrix is called the characteristic adjoint matrix $O(\lambda)$

$$P(\lambda) = \det(\lambda I - A) = k_0\lambda^n + k_1\lambda^{n-1} + \ldots + k_{n-1}\lambda + k_n$$

$$Q(\lambda) = \operatorname{adj}(\lambda I - A) = \lambda^{n-1}B_0 + \lambda^{n-2}B_1 + \ldots + \lambda B_{n-2} + B_{n-1}$$

$$k_0 = 1, \quad k_n = (-1)^n \det A, \quad B_0 = I, \quad B_{n-1} = (-1)^{n-1}\operatorname{adj} A$$

The algorithm, which is an improvement of Le Verrier's method [5] and which enables one to calculate simultaneously the scalars k_i and the matrices B_i, was proposed by Souriau in 1948 and verified on the first computers in the USA. Later this algorithm was sometimes attributed to Faddeev [6] or Frame [7].

The algorithm is based on the property $A \operatorname{adj} A = I \det A$, which relates the adjoint matrix and determinant, and on the relation between the derivative of the determinant and the characteristic adjoint matrix (a prime denotes a derivative with respect to λ)

$$(\lambda I - A)Q(\lambda) = P(\lambda)I, \quad P'(\lambda) = \operatorname{tr} Q(\lambda) \tag{2.1}$$

Equating the coefficients of like powers of λ on the left and right sides of relations (2.1), we arrive at the following recurrence formulae [2] of the algorithm

$$B_0 = I, \quad A_i = AB_{i-1}, \quad k_i = \frac{\operatorname{tr} B_i}{n-i}, \quad B_i = A_i + k_i I; \quad i = 1, \ldots, n$$

3. SOLUTION OF THE SYSTEM OF DIFFERENTIAL EQUATIONS WITH FIRST-ORDER DERIVATIVES

Suppose the linear matrix ordinary differential equation has the form

$$\dot{x} = Ax \tag{3.1}$$

To construct a solution we calculate the coefficients k_i of the characteristic polynomial and the matrix coefficients B_i of the characteristic adjoint matrix. The first of relations (2.1) is the key one in Souriau's algorithm. Substituting the operator d/dt instead of the variable λ, we obtain

$$\left(I\frac{d}{dt} - A\right)Q\left(\frac{d}{dt}\right) = IP\left(\frac{d}{dt}\right)$$

and suppose $\gamma(t)$ is a particular scalar solution of the ordinary differential equation

$$P\left(\frac{d}{dt}\right)\gamma(t) = \sum_{i=0}^{n} k_i\gamma^{(n-i)} = 0, \quad \gamma^{(n-i)} = \frac{d^{n-i}}{dt^{n-i}}\gamma(t) \tag{3.2}$$

Then the matrix function

$$\phi(t) = Q\left(\frac{d}{dt}\right)\gamma(t) = \sum_{i=0}^{n-1} B_i\gamma^{(n-i-1)}(t) \tag{3.3}$$

is a solution of the matrix ordinary differential equation

$$\dot{\phi} = A\phi(t)$$

By virtue of this property, the linear ordinary differential equation (3.2) will be called the characteristic ordinary differential equation for ordinary differential equation (3.1).

In order that the matrix $\phi(t)$ should be exactly equal to the exponential function e^{At}, it is necessary that $\phi(0) = I$. By virtue of the equality $B_0 = I$ and Eq. (3.3), to solve the linear ordinary differential equation (3.2) we must choose the initial conditions

$$\gamma^{(n-2)}(0) = 1, \quad \gamma^{(n-1)}(0) = \ldots = \gamma^{(1)}(0) = \gamma(0) = 0 \tag{3.4}$$

With this choice of the initial conditions for the function $\gamma(t)$ the general solution of ordinary differential equation (3.1) can be written in the form

$$x(t) = \phi(t)x_0, \quad x(0) = x_0 \tag{3.5}$$

The solution $\gamma(t)$ of characteristic ordinary differential equation (3.2) is satisfactorily obtained only in the part $[0, h]$ of the specified interval of integration. In the remaining part, the exponential function $\phi(t)$ is calculated using matrix multiplication, based on the property

$$\phi(pt) = \phi^p(t) \tag{3.6}$$

Thus, the algorithm for obtaining a general solution of a system of ordinary differential equations of the form (3.1) consists of the following steps.

1. Calculate k_i and B_i using Souriau's algorithm.
2. Calculate the function $\gamma(t)$ by integrating ordinary differential equation (3.2) with initial conditions (3.4) in a small time interval $[0, h]$.
3. Tabulate the matrix function φ using formula (3.3) in the time interval $[0, h]$.
4. Extend the tabulation region 3 to $[0, ph]$ using property (3.6).
5. Tabulate the solution (3.5).

This algorithm can easily be extended to ordinary differential equations of the form $A\dot{x} + Bx = 0$ with characteristic matrix $\lambda A + B$.

4. FREE OSCILLATIONS OF MECHANICAL SYSTEMS

The ordinary differential equation of free oscillations of a linear dissipative mechanical system has the form

$$M\ddot{x} + C\dot{x} + Kx = 0 \tag{4.1}$$

with specified initial coordinates and velocities x_0 and $\dot{x}_0$. The matrices M, C and K do not necessarily satisfy the usual properties of symmetry and sign-definiteness [8, 9].

Two methods of reducing the second-order matrix ordinary differential equation (4.1) to a first-order matrix ordinary differential equation of double dimensionality are known. In the first method, after introducing the additional variables $y = \dot{x}$, ordinary differential equation (4.1) can be reduced to an ordinary differential equation of the form [10, 11]

$$\dot{z} = Az, \quad z = \|x^T, y^T\|^T \tag{4.2}$$

To obtain the $2n \times 2n$ matrix A it is necessary to invert the matrix of the masses M. For simple eigenvalues of the matrix A, the solution of ordinary differential equation (4.2) can be represented by a linear combination of solutions of the form $e^{\lambda_k t}\upsilon_k$, where λ_k are the eigenvalues of the matrix A and υ_k are the eigenvectors corresponding to them. In the case of multiple eigenvalues there will additionally be polynomial coefficients in t. In the second method, ordinary differential equation (4.1) is reduced to the form

$$A\dot{z} + Bz = 0$$

where A and B are $2n \times 2n$ matrices. The advantage of this method is the fact that the matrices A and B turn out to be symmetrical and inversion of the matrix M is not required.

In this section the method developed in Section 3 for first-order linear matrix ordinary differential equations is extended to a second-order ordinary differential equation of the form (4.1) without reduction to first-order equations.

Just as in Section 2, the matrix $\lambda^2 M + \lambda C + K$ will be called the characteristic matrix, its determinant $P(\lambda)$ will be called the characteristic polynomial, and the transposed matrix of its cofactors $Q(\lambda)$ will be called the characteristic adjoint matrix

$$P(\lambda) = \det(\lambda^2 M + \lambda C + K) = \sum_{i=0}^{2n} k_i \lambda^{2n-i}$$

$$Q(\lambda) = \operatorname{adj}(\lambda^2 M + \lambda C + K) = \sum_{i=0}^{2n-2} B_i \lambda^{2n-2-i}$$

$$k_0 = \det M, \quad k_{2n} = \det K, \quad B_0 = \operatorname{adj} M, \quad B_{2n-2} = \operatorname{adj} K$$

The quantities k_0 and B_0 will be used as the initial conditions for the recurrence relations of Souriau's algorithm, adapted for solving ordinary differential equation (4.1).

We will start by considering the algebraic part of the method, which we will divide into three steps. At the first step we use an identity similar to the first identity in formula (2.1),

$$(\lambda^2 M + \lambda C + K) Q(\lambda) = P(\lambda) I \tag{4.3}$$

Equating coefficients of like powers of λ we arrive at $2n + 1$ relations between the coefficients k_i and B_i

$$\begin{aligned} &MB_0 = k_0 I, \quad MB_1 + CB_0 = k_1 I, \\ &MB_2 + CB_1 + KB_0 = k_2 I, \ldots, MB_{2n-2} + CB_{2n-3} + KB_{2n-4} = k_{2n-2} I, \\ &CB_{2n-2} + KB_{2n-3} = k_{2n-1} I, \quad KB_{2n-2} = k_{2n} I \end{aligned} \tag{4.4}$$

At the second step, by calculating the trace of the matrices on the left- and right-hand sides of relation (4.4), we obtain the following $2n + 1$ scalar relations

$$\begin{aligned} &\operatorname{tr}(MB_0) = nk_0, \quad \operatorname{tr}(MB_1) + \operatorname{tr}(CB_0) = nk_1 \\ &\operatorname{tr}(MB_2) + \operatorname{tr}(CB_1) + \operatorname{tr}(KB_0) = nk_2, \ldots \\ &\ldots, \operatorname{tr}(CB_{2n-2}) + \operatorname{tr}(KB_{2n-3}) = nk_{2n-1}, \quad \operatorname{tr}(KB_{2n-2}) = nk_{2n} \end{aligned} \tag{4.5}$$

At the third step, using the property of the characteristic matrix

$$P'(\lambda) = \frac{d}{d\lambda}(\det(\lambda^2 M + \lambda C + K)) = \operatorname{tr}(Q(\lambda)(2\lambda M + C))$$

we obtain

$$\begin{aligned} &2nk_0\lambda^{2n-1} + (2n-1)k_1\lambda^{2n-2} + \ldots + 2k_{2n-2}\lambda + k_{2n-1} = \\ &= \operatorname{tr}(2MB_0)\lambda^{2n-1} + \operatorname{tr}(2MB_1 + CB_0)\lambda^{2n-2} + \ldots + \\ &+ \operatorname{tr}(2MB_{2n-2} + CB_{2n-3})\lambda + \operatorname{tr}(CB_{2n-3}) \end{aligned} \tag{4.6}$$

Equating coefficients of like powers of λ in relation (4.6), we obtain

$$\begin{aligned} &2nk_0 = 2\operatorname{tr}(MB_0), \quad (2n-1)k_1 = 2\operatorname{tr}(MB_1) + \operatorname{tr}(CB_0), \ldots \\ &\ldots, 2k_{n-2} = 2\operatorname{tr}(MB_{n-2}) + \operatorname{tr}(CB_{2n-3}), \quad k_{2n-1} = \operatorname{tr}(CB_{2n-2}) \end{aligned} \tag{4.7}$$

Comparing equalities (4.4), (4.5) and (4.7), to calculate the coefficients k_i and B_i for the specified parameters n, M, C and K we obtain the following recurrence relations for $i = 2, \dots, 2n$

$$k_i = (\mathrm{tr}(CB_{i-1}) + 2\mathrm{tr}(KB_{i-2}))/i, \quad B_i = B_0(k_iI - CB_{i-1} - KB_{i-2})/k_0$$

with initial conditions

$$k_0 = \det M, \quad k_1 = \mathrm{tr}(CB_0), \quad B_0 = \mathrm{adj}M, \quad B_1 = B_0(k_1I - CB_0)/k_0$$

Here, for brevity, we have supplemented the zero matrices $B_{2n-1} = B_{2n} = 0$.

We will now present a numerical computation of the solutions $\gamma(t)$. In algebraic identity (4.3) we replace λ by d/dt and compute the numerical solution $\gamma(t)$ of the ordinary differential equation

$$P\left(\frac{d}{dt}\right)\gamma = k_0\gamma^{(2n)} + k_1\gamma^{(2n-1)} + k_2\gamma^{(2n-2)} + \dots + k_{2n-2}\gamma^{(2)} + k_{2n-1}\gamma^{(1)} + k_{2n}\gamma = 0 \tag{4.8}$$

Then the matrix

$$\phi(t) = Q\left(\frac{d}{dt}\right)\gamma = B_0\gamma^{(2n-2)} + B_1\gamma^{(2n-3)} + \dots + B_{2n-3}\gamma^{(1)} + B_{2n-2}\gamma$$

will satisfy the ordinary differential equation

$$M\ddot{\phi} + C\dot{\phi} + K\phi = 0 \tag{4.9}$$

We will represent the general solution of ordinary differential equation (4.1) by a linear combination of two linearly independent solutions, which can be obtained by choosing two sets of initial conditions (for $k = 1$ and $k = 2$) for $\gamma(t)$,

$$\gamma_k^{(2n-3+k)}(0) = 1, \quad \gamma_k^{(2n-k)}(0) = \gamma_k^{(2n-3)}(0) = \dots = \gamma_k^{(1)}(0) = \gamma_k(0) = 0 \tag{4.10}$$

i.e. the initial data is both sets are zeros, with the exception of unities for the leading derivative of order $(2n-2)$ in the first set and for the derivative of order $(2n-1)$ in the second set. Then the matrix solutions of ordinary differential equation (4.9)

$$\phi_1(t) = Q\left(\frac{d}{dt}\right)\gamma_1; \quad \phi_2(t) = Q\left(\frac{d}{dt}\right)\gamma_2$$

satisfy the initial conditions

$$\phi_1(0) = B_0, \quad \phi_1^{(1)}(0) = B_1, \quad \phi_2(0) = 0, \quad \phi_2^{(1)}(0) = B_0 \tag{4.11}$$

The solution of ordinary differential equation (4.1) can be represented in the form

$$x(t) = \phi_1(t)v_1 + \phi_2(t)v_2$$

where the constant vectors v_1 and v_2 are found from the equations

$$B_0v_1 = x_0, \quad B_1v_1 + B_0v_2 = \dot{x}_0$$

Taking the relation $B_0 = \mathrm{adj}M$ into account, we obtain for v_1 and v_2

$$v_1 = \frac{1}{k_0}Mx_0, \quad v_2 = \frac{1}{k_0}M(\dot{x}_0 - B_1v_1) \tag{4.12}$$

The solution $x(t)$ was obtained in the time interval $[0, h]$. Just as in the case of the first-order matrix ordinary differential equation, this solution can be continued to the time interval $[0, ph]$ for any integer p using the algebraic relation

$$\left(\frac{1}{k_0}\Phi(t)W\right)^p = \frac{1}{k_0}\Phi(pt)W(t)$$

$$\Phi(t) = \begin{Vmatrix} \phi_1(t) & \phi_2(t) \\ \phi_1^{(1)}(t) & \phi_2^{(1)}(t) \end{Vmatrix}, \quad W = \begin{Vmatrix} M & 0 \\ -\frac{1}{k_0}MB_1M & M \end{Vmatrix} \tag{4.13}$$

Thus, the algorithm for obtaining a general solution of ordinary differential equation (4.1) consists of the following steps.

1. Calculate k_i and B_i using Souriau's algorithm, adapted for an equation of the form (4.1).
2. Find the functions $\gamma_1(t)$ and $\gamma_2(t)$ by numerical integration of ordinary differential equation (4.8) with initial conditions (4.10) in the short time interval $[0, h]$.
3. Tabulate the matrix functions $\phi_1(t)$ and $\phi_2(t)$ in the time interval $[0, h]$.
4. Extend the tabulation 3 to the time interval $[0, ph]$ using formula (4.13).
5. Calculate v_1 and v_2 from formula (4.12).
6. Tabulate the solution $x(t) = \phi_1(t)v_1 + \phi_2(t)v_2$.

5. FORCED OSCILLATIONS OF MECHANICAL SYSTEMS

Forced oscillations of a linear dissipative mechanical system with n degrees of freedom are described by ordinary differential equation (1.1) with a time-dependent right-hand side. We obtain a solution by the method of Lagrange variation of the arbitrary constants v_1 and v_2, considering them as functions of time

$$x(t) = \phi_1(t)v_1(t) + \phi_2(t)v_2(t) \tag{5.1}$$

Here we use the algorithm from Section 4 up to step 3, in which the functions ϕ_1 and ϕ_2 are determined.

As usual, we connect the derivatives v_1 and v_2 by the additional condition

$$\phi_1\dot{v}_1 + \phi_2\dot{v}_2 = 0 \tag{5.2}$$

Then

$$\dot{x}(t) = \dot{\phi}_1(t)v_1(t) + \dot{\phi}_2(t)v_2(t) \tag{5.3}$$

Differentiating Eq. (5.2), we obtain

$$\phi_1\ddot{v}_1 + \phi_2\ddot{v}_2 = -\dot{\phi}_1\dot{v}_1 - \dot{\phi}_2\dot{v}_2$$

Putting $t = 0$ in relations (5.1) and (5.3) and taking the initial conditions (4.11) into account, we obtain two relations between the initial data for v_1, v_2 and x

$$x_0 = B_0v_1(0), \quad \dot{x}_0 = B_1v_1(0) + B_0v_2(0) \tag{5.4}$$

From this we obtain

$$v_1(0) = \frac{1}{k_0}Mx_0, \quad v_2(0) = \frac{1}{k_0}M(\dot{x}_0 - B_1v_1) \tag{5.5}$$

To determine the functions $v_1(t)$ and $v_2(t)$ it remains to integrate the following system of linear ordinary differential equations

$$\phi_1\dot{v}_1 + \phi_2\dot{v}_2 = 0, \quad M\dot{\phi}_1\dot{v}_1 + M\dot{\phi}_2\dot{v}_2 = F(t) \tag{5.6}$$

or in partitioned matrix form

$$G\dot{v} = \Phi; \quad G = \begin{Vmatrix} \phi_1 & \phi_2 \\ M\dot{\phi}_1 & M\dot{\phi}_2 \end{Vmatrix}, \quad v = \begin{Vmatrix} \dot{v}_1 \\ \dot{v}_2 \end{Vmatrix}, \quad \Phi = \begin{Vmatrix} 0 \\ F(t) \end{Vmatrix} \tag{5.7}$$

Using the adjoint matrix and the determinant, we obtain

$$\dot{\upsilon} = \frac{\mathrm{adj}G}{\det G}\Phi \tag{5.8}$$

or, after integration,

$$\upsilon(t) = \upsilon(0) + \int_0^t \frac{\mathrm{adj}G(s)}{\det G(s)}\Phi ds \tag{5.9}$$

Finally, the solution of ordinary differential equation (1.1) has the form

$$x(t) = \|\phi_1(t), \phi_2(t)\| \upsilon(t) \tag{5.10}$$

The effectiveness of the proposed algorithms was verified by investigating actual linear oscillatory systems with symmetrical and asymmetrical matrices of the coefficients.

REFERENCES

1. SOURIAU, J.-M., Une méthode par la décomposition spectrale et l'inversion des matrices. *C.-R. Acad. Sci.*, 1948, **227**, 2, 1010–1011.
2. SOURIAU, J.-M., *Calcul Linéaire*, Vol. 1, Presses Univers. France, Paris. 1964.
3. RÉAUD K., SOURIAU, J.-M., VALLÉE, C. L. and FORTUNÉ, D., Méthode de Le Verrier-Souriau et équations différentielles linéaires. *C.-R. Acad. Sci. Paris, Sér. IIb.*, 2000, **328**, 10, 773–778.
4. MA, F. and CAUGHEY, T. K., Analysis of linear nonconservative vibrations. *Trans. ASME J. Appl. Mech.*, 1995, **62**, 3, 685–691.
5. LE VERRIER, U., Variations séculaires des éléments elliptiques des sept planétes principales: Mercure, Venus, la Terre, Mars, Jupiter, Saturne et Uranus. *J. Math. Pures Appli.*, 1840, 4, 220–254.
6. FADDEEV, D. K. and FADDEEVA, F. N., *Computational Methods of Linear Algebra*. W. H. Freeman and Company, San Francisco and London, 1963.
7. FRAME, J. S., A simple recursion formula for inverting a matrix (abstract). *Bull. Amer. Math. Soc.*, 1949, 55, 1045.
8. GANTMACHER, F. R., *The Theory of Matrices*. Vol. 1. Chelsea, New York, 1959.
9. HOUSEHOLDER, A. S., *The Theory of Matrices in Numerical Analysis*. Blaisdell, New York, 1964.
10. INMAN, D. J., Dynamics of asymmetric nonconservative systems. *Trans. ASME J. Appl. Mech.*, 1983, **50**, 1, 199–203.
11. FAWZY, I. and BISHOP, R. E. D., On the dynamics of linear non-conservative systems. *Proc. Roy. Soc. London: Ser. A.*, 1976, **352**, 1668, 25–40.

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